Improved efficiency and accuracy using duality in hybrid boundary element-surface impedance boundary condition formulation.

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The paper present a hybrid formulation coupling the surface impedance boundary condition to deal with conductive and magnetic materials and the boundary element method to account rigorously for the regularity conditions of the magnetic field at infinity. Based on duality relations, the field variables are associated to spatial elements starting from the dual mesh, overcoming some of the performance issues of the previously published results. The new formulation showed also a better accuracy in the calculation of postprocessing quantities.

Index Terms—Boundary element method, duality, hybrid formulation, surface impedance boundary condition.

I. INTRODUCTION

IN RECENT papers [1], [2] the possibility of coupling nonlinear surface impedance boundary conditions (SIBC) with the boundary element method (BEM) was presented. The BEM-SIBC hybrid formulation was based on the definition of integral variables over two intertwined surface meshes, namely primal and dual meshes, linked by duality relations. This particular choice gave rise to a lumped circuit network of admittances in the conductive domain which was coupled with the boundary element formulated in terms of reduced scalar potential and its normal derivative. The method showed the benefits of the two formulations: the SIBC allowed the use of analytical or semi-analytical solution of the the electromagnetic field in thin conductive layers [3], also in case of nonlinear material, while the BEM rigorously accounted for the regularity conditions of the magnetic field at infinity. The main disadvantage of the method was the definition of the SIBC equations in terms of the admittance matrix $\mathbf{Y}$, that required a direct LU factorization in order to obtain the impedance matrix $\mathbf{Z}$ necessary for the solution of the final system. This drawback was even more amplified in case of nonlinear iterations. Despite some strategies for the direct construction of the matrix $\mathbf{Z}$ on the primal mesh are available in the literature, like in [4], these technique are not consistent [5]. We propose to solve this problem by swapping the variable association between the primal and dual mesh, building the matrix $\mathbf{Z}$ directly on the dual mesh.

II. SPACE DISCRETIZATION AND CONVENTIONAL SIBC-BEM

The domain under study is constituted by conductive regions, where eddy currents occur, external sources and external open space. It is assumed that the combination of the material properties and the source frequency is such that the penetration depth δ can be considered small with respect to the conductive region dimension. Under this hypothesis, it is possible to discretize only the surface of the domain by an unstructured mesh, referred to as simplicial mesh. Starting from this mesh, a second based on barycentric subdivision is derived. This second
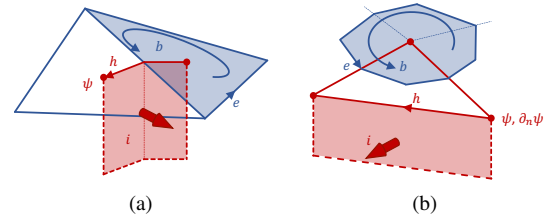

Fig. 1: Definition of variables: (a) conventional association (b) dual association.

discretization is referred to as barycentric mesh, Fig. 1a. The electric voltages e and the magnetic fluxes b are associated to the edges and the faces of the simplicial mesh, respectively. The magnetic scalar potential ψ and its normal derivative $\partial_n\psi$, the magneto-motive force and the electric current are defined on the nodes, edges and faces of the barycentric mesh. Using the Ampère's and Faraday's laws, imposing the continuity of the tangential component of the magnetic field and the normal component of the magnetic flux density and coupling with the BEM equation, the final system is

$$\begin{bmatrix} -\mathbf{H} & \mathbf{W} \\ \mathbf{C}\mathbf{Y}^{-1}\mathbf{C}^{\mathrm{T}} & \mathrm{j}\omega\mu_0\mathbf{S} \end{bmatrix} \begin{bmatrix} \psi \\ \partial_n\psi \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix} \quad (1)$$

where: $\mathbf{H}$ and $\mathbf{W}$ are the standard matrices that arise from the boundary element method formulated in terms of reduced magnetic scalar potential ψ and its normal derivative $\partial_n\psi$, $\mathbf{C}$ is the primal curl matrix linking triangular faces to their bounding edges, $\mathbf{Y}$ is the admittance linking voltage to current, $\mathbf{S}$ is the area matrix of each triangular facet. The right hand side $\mathbf{b} = \mathbf{C}\mathbf{Y}^{-1}\mathbf{h}_{\mathrm{S}} + \mathrm{j}\omega\mu_0\mathbf{S}\mathbf{H}_{\mathrm{S}n}$, where $\mathbf{H}_{\mathrm{S}n}$ and $\mathbf{h}_{\mathrm{S}}$ are contributes of source currents on faces and along primal edges. Details of the formulation can be found in [2]. The unknowns of the systems are the N_F scalar potentials ψ and their N_F normal derivatives $\partial_n\psi$ defined on barycentric nodes.

III. DUAL SIBC-BEM

The previously described problem is re-formulated by swapping the association of the variables with the space elements.

This time the electric voltages are associated to the edges and the magnetic fluxes to the faces of the barycentric mesh, whereas the scalar potentials and their normal derivatives, the magneto-motive forces and the currents are associated respectively to the simplicial nodes, edges and faces, Fig. 1b. In terms of matrix operators, Faraday's law becomes:

$$\tilde{\mathbf{C}}\mathbf{e} = \mathrm{j}\omega\mathbf{b} \tag{2}$$

where, in 2-dimensional discretization, the discrete curl operator $\tilde{\mathbf{C}} = \mathbf{G}^{\mathrm{T}}$. The Ampère's equation reads $\mathbf{h} = \mathbf{i}$. The constitutive equation links the voltages to the currents. The electric voltage can be calculated by integration of the electric field along dual edges:

$$e'_j = \int_{\tilde{L}_j} \vec{E}_0 \cdot \vec{t}\mathrm{d}L = \int_{\tilde{L}_j} \frac{1}{\sigma}\vec{J}_0 \cdot \vec{t}\mathrm{d}L \tag{3}$$

where the current density $\vec{J}(z)$ is a function of the depth coordinate z and is expressed as:

$$\vec{J}(z) = \vec{J}_0 f(z) = \vec{J}_0 \exp^{-(1+\mathrm{j})\frac{z}{\delta}} \tag{4}$$

Expanding the current density using facet elements $\vec{w}_m$, the surface current density becomes:

$$\vec{J}_0 = \frac{\sum_{m=1}^{3} \vec{w}_m i_m}{\int_0^\infty f(z)\mathrm{d}z} = \frac{1+\mathrm{j}}{\delta} \sum_{m=1}^{3} \vec{w}_m i_m \tag{5}$$

Thus, (6) becomes

$$e'_j = \sum_{m=1}^{3} i_m \frac{1+\mathrm{j}}{\sigma\delta} \int_{\tilde{L}_j} \vec{w}_m \cdot \vec{t}\mathrm{d}L = \sum_{m=1}^{3} Z'_{jm} i_m \tag{6}$$

Assembling element-by-element the impedance matrix $\mathbf{Z}$, the voltage-current relation becomes $\mathbf{e} = \mathbf{Z}\mathbf{i}$

A. Interface conditions and BEM formulation

The interface conditions are used to imposed the continuity of the tangential component of the magnetic filed and the normal component of the magnetic flux density. Similarly to the conventional SIBC-BEM formulation, these equations are:

$$\mathbf{h} = -\mathbf{G}\boldsymbol{\psi} + \mathbf{h}_{\mathrm{S}} \tag{7}$$

$$\mathbf{b} = \mu_0 \mathbf{S}(-\partial_n\boldsymbol{\psi} + \mathbf{H}_{\mathrm{S}n}) \tag{8}$$

but the matrix is not diagonal anymore.

Because the scalar potentials and their normal derivative are defined on the nodes of the simplicial mesh, linear elements are used. This requires the use of nodal functions N and the calculation of the integrals of the Green's function $G(\vec{r}_i, \vec{r}_j)$:

$$\{\mathbf{H}\}_{ij} = \delta_{ij}\alpha - \int_{S_j} N_i \frac{\partial G(\vec{r}_i, \vec{r}_j)}{\partial n}\mathrm{d}S \tag{9}$$

$$\{\mathbf{W}\}_{ij} = \int_{S_j} N_i G(\vec{r}_i, \vec{r}_j)\mathrm{d}S \tag{10}$$

The additional burden with respect to the BEM formulation with constant elements is usually negligible when using analytical formulas [6].
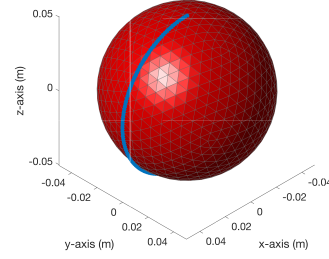

Fig. 2: Benchmark sphere and sampling points. The external field B_0 is aligned with the z axis.
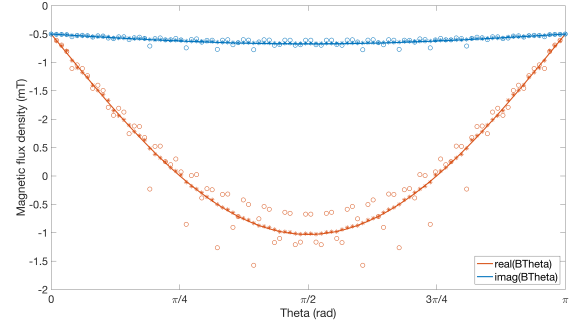

Fig. 3: Polar component of the magnetic flux density in proximity of the sphere surface. Line: analytical solution, circle: conventional SIBC-BEM, star: dual SIBC-BEM.

IV. RESULTS

The two formulations are compared against the analytical solution of a sphere of radius $R_0 = 5$ cm in a uniform field $B0 = 1$ mT at 1 kHz. The sphere is characterized by $\sigma = 10$ MS/m and $\mu_R = 1$. The magnetic field is measured close to the sphere surface, Fig. 2. Fig. 3 shows the polar component of the magnetic flux density. The dual SIBC-BEM shows a better accuracy due to the use to higher order BEM elements. In addition the number of unknowns is almost halved, since the number of triangles in a surface mesh is approximately twice the number of nodes. In the full paper details about the numerical formulation and the convergence analysis of the two methods will be provided.

REFERENCES

[1] V. Cirimele, F. Freschi, L. Giaccone, and M. Repetto, "Finite formulation of surface impedance boundary conditions," *IEEE Transactions on Magnetics*, vol. 52, no. 3, pp. 7 206 504/1–7 206 504/4, 2016.

[2] F. Freschi, L. Giaccone, and M. Repetto, "Nonlinear bem-surface impedance boundary condition formulation for unstructured meshes," in *Proceedings of 17th Biennial IEEE Conference on Electromagnetic Field Computation*, 2016.

[3] B. Ancelle, A. Nicolas, and J. Sabonnadiere, "A boundary integral equation method for high frequency eddy currents," *IEEE Transactions on Magnetics*, vol. 17, no. 6, pp. 2568–2570, 1981.

[4] P. Alotto, A. De Cian, and G. Molinari, "A time-domain 3-D full-Maxwell solver based on the Cell Method," *IEEE Transactions on Magnetics*, vol. 42, no. 4, pp. 799–802, 2006.

[5] L. Codecasa, R. Specogna, and F. Trevisan, "Discrete constitutive equations over hexahedral grids for eddy-current problems," *Computer Modeling in Engineering & Sciences*, vol. 31, no. 3, pp. 129–144, 2008.

[6] R. Graglia, "On the numerical integration of the linear shape functions times the 3-D Green's function or its gradient on a plane triangle," *IEEE Transactions on Antennas and Propagation*, vol. 41, pp. 1448–1455, 1993.